Analyzing Functions

Implicit Functions and Implicit Differentiation

In mathematics, an **implicit function** is a generalization of the concept of a function in which the dependent variable, say, y has not been given "explicitly" in terms of the independent, say, x. To give a function **explicitly** is to be able to express it as $y = f(x)$. By contrast, the function is **implicit** if the value of y is obtained from x by solving an equation of the form $R(x, y) = 0$.

Definition of an Implicit Function

Consider the equation $R(x, y) = 0$. Then
y is an <u>implicit function</u> of x in $R(x, y) = 0 \overset{def}{\equiv}$ if there exists a function $y = f(x)$ on (a, b) such that $R(x, f(x)) = 0$, $\forall x \in (a, b)$.

An implicit function such as $R(x, y) = 0$ can be a useful way to express a functional relationship that may be too complicated, inconvenient or even impossible to solve for y in terms of x. For example, the implicit function $y + e^y = x^5$ would be impossible to solve for y in terms of x. In another situation, the equation $R(x, y) = 0$ may fail to define a function at all and would express a multi-valued function. For example, the equations $4\left(x^2 + y^2 - ax\right)^3 = 27a^2\left(x^2 + y^2\right)^2$, called the Cayley's Sextic, (see Figure 1) and $\left(x^2 + y^2 + 12ax + 9a^2\right)^2 = 4a(2x + 3a)^3$, called the Tricuspoid (see Figure 2), express such multi-valued functions, that would be too difficult, if not impossible, to solve for y in terms of x.

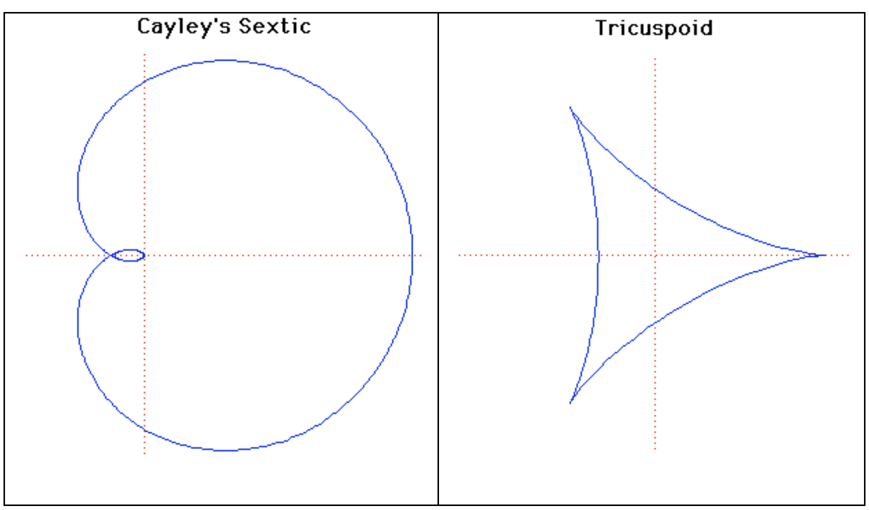

Figure 1 Figure 2

In spite of the difficulties in expressing an implicit function explicitly, it is relatively easy to determine its derivative $\frac{dy}{dx}$. As an example, determine $\frac{dy}{dx}$ implicitly, if $xy = \sin y + x^2 y^2$. In determining $\frac{dy}{dx}$ implicitly, you must think " y is a function of x ". To emphasize this point, let us suppose that there exists a function $y = f(x)$ that satisfies the implicit function $xy = \sin y + x^2 y^2$. That is,

$x \cdot f(x) = \sin(f(x)) + x^2 \cdot [f(x)]^2$. Then, we wish to find $f'(x)$. Differentiating both sides with respect to x, we obtain

$$
\begin{aligned}
& x \cdot f(x) = \sin(f(x)) + x^2[f(x)]^2 \Rightarrow \\
& x \cdot f'(x) + f(x) = \cos(f(x)) \cdot f'(x) + x^2 \cdot 2 \cdot f(x) \cdot f'(x) + 2 \cdot x \cdot [f(x)]^2 \Rightarrow \\
& \left(x - \cos(f(x)) - 2x^2 \cdot f(x)\right) f'(x) = 2x \cdot [f(x)]^2 - f(x) \Rightarrow \\
& f'(x) = \frac{2x \cdot [f(x)]^2 - f(x)}{\left(x - \cos(f(x)) - 2x^2 \cdot f(x)\right)}
\end{aligned}
$$

Exercises: Find $\frac{dy}{dx}$ implicitly if

1. $4\left(x^2 + y^2 - ax\right)^3 = 27a^2\left(x^2 + y^2\right)^2$.
2. $y^2 = x^2(a - x)/(a + x)$.
3. $\left(x^2 + y^2 + 12ax + 9a^2\right)^2 = 4a(2x + 3a)^3$.

Rate of Change

The classic example of a rate of change is that of velocity in rectilinear motion. Consider some object whose motion is on the real line and whose position s at any time t from the origin is given by the function $s = f(t)$, called the equation of motion. We are all familiar with the concept of an average velocity in which we compare the change in position Δs with the corresponding change in time Δt by dividing the former by the latter. Thus, if we consider the motion of our object during an interval of time $[t, t+\Delta t]$ or $[t+\Delta t, t]$, we obtain the following definitions[1]:

Average Velocity $\bar{v}$

Let $s = f(t)$ on $[0,b]$
The average velocity $\bar{v}$ during $[t, t+\Delta t] \overset{def}{\equiv} \Delta s = f(t+\Delta t) - f(t) \Rightarrow$ $\bar{v} = \dfrac{\Delta s}{\Delta t} = \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$.

From this average, we obtain the concept of an instantaneous velocity at t by taking the limit of this average as $\Delta t \to 0$.

Instantaneous Velocity v

Let $s = f(t)$ on $[0,b]$
The instantaneous velocity v at $t \overset{def}{\equiv} \Delta s = f(t+\Delta t) - f(t) \Rightarrow$ $v = \dot{s} = \dfrac{ds}{dt} = f'(t) = \lim_{\Delta t \to 0} \bar{v} = \lim_{\Delta t \to 0} \dfrac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$, provided this limit exists.

In the same way, we first define the average acceleration $\bar{a}$ during $[t, t+\Delta t]$. Then, from this average, we obtain the concept of an instantaneous acceleration a at t by taking the limit of this average as $\Delta t \to 0$.

Instantaneous Acceleration a

Let $v = f'(t)$ on $[0,b]$
The instantaneous acceleration a at $t \overset{def}{\equiv} \Delta v = f'(t+\Delta t) - f'(t) \Rightarrow \bar{a} = \dfrac{\Delta v}{\Delta t} \Rightarrow$ $a = \ddot{s} = \dfrac{d^2 s}{dt^2} = f''(t) = \lim_{\Delta t \to 0} \bar{a} = \lim_{\Delta t \to 0} \dfrac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \dfrac{f'(t+\Delta t) - f'(t)}{\Delta t}$, provided this limit exists.

In comparing two quantities, we do not have to restrict ourselves to position versus time. Realizing that we can compare any two quantities and taking our motivation from the concept of velocity, we can abstract this latter concept of velocity and, thus, derive the concept of an instantaneous rate of change of one quantity with respect to another quantity. Toward this end, consider two quantities q, Q functionally related by $Q = f(q)$.

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[1] We must allow both $\Delta t > 0$ and $\Delta t < 0$, when considering averages and limits. However, to save space we will usually only illustrate the case in which $\Delta t > 0$.

Average Rate of Change $\overline{Q}$ of the Quantity Q

Let $Q = f(q)$ on $[a, b]$
The average rate of change of the quantity $\overline{Q}$ with respect to the quantity q during $[q, q + \Delta q] \overset{def}{\equiv} \Delta Q = f(q + \Delta q) - f(q) \Rightarrow$ $\overline{Q} = \dfrac{\Delta Q}{\Delta q} = \dfrac{f(q + \Delta q) - f(q)}{\Delta q}$.

Instantaneous Rate of Change of the Quantity Q

Let $Q = f(q)$ on $[a, b]$
The instantaneous rate of change $\dfrac{dQ}{dq}$ of the quantity Q with respect to the quantity q at some $q \overset{def}{\equiv} \Delta Q = f(q + \Delta q) - f(q) \Rightarrow$ $\dfrac{dQ}{dq} = f'(q) = \lim\limits_{\Delta q \to 0} \overline{Q} = \lim\limits_{\Delta q \to 0} \dfrac{\Delta Q}{\Delta q} = \lim\limits_{\Delta q \to 0} \dfrac{f(q + \Delta q) - f(q)}{\Delta q}$, provided this limit exists.

Examples:

1) The surface area A of a sphere of radius r is $A = 4\pi r^2$.
   a) What is the rate of change[2] of the surface of a sphere with respect to its radius r.
   b) What is the rate of change when $r = 3$?
   c) How must r be chosen if the rate of change is 1?

   Answers:

$$A = 4\pi r^2 \Rightarrow \frac{dA}{dr} = 8\pi r.$$

(a) $\left.\dfrac{dA}{dr}\right|_{r=3} = \left.8\pi r\right|_{r=3} = 24\pi$

(b) $\dfrac{dA}{dr} = 8\pi r \Rightarrow 1 = \dfrac{dA}{dr} = 8\pi r \Rightarrow r = \dfrac{1}{8\pi}$.

2) For what value of x is the rate of change of $y = ax^2 + bx + c$ with respect to x the same as the rate of change of $z = bx^2 + ax + c$ with respect to x? Assume a, b, c are constants with $a \neq b$.

   Answer: We must have: $\dfrac{dy}{dx} = \dfrac{dz}{dx} \Rightarrow 2ax + b = 2bx + a \Rightarrow 2(a - b)x = a - b \Rightarrow x = \dfrac{a - b}{2(a - b)} = \dfrac{1}{2}$. Thus, $x = \dfrac{1}{2}$.

[2] We will frequently use the phrase "rate of change" without modifier to mean "instantantaneous rate of change".

Exercises:

1) Car A traveling west at 30 mph passes intersection P at noon while car B traveling south at 40 mph passes intersection P three hours later. How fast is the distance between the cars changing at 4:00 p.m.?
2) A spot light is on the ground 20 ft away from a wall and a 6 ft tall person is walking towards the wall (See Figure 3). What is the rate of change of the height of the shadow with respect to the person's distance from the spotlight at the instant the person is 8 feet from the wall?

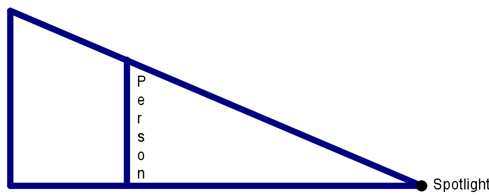
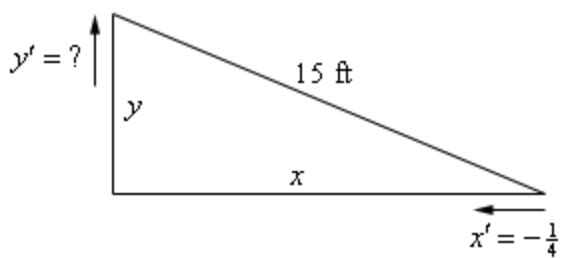


Figure 3

3) A 15 foot ladder is resting against the wall (See Figure). The bottom is initially 10 feet away from the wall and is being pushed towards the wall. How fast is the top of the ladder moving up the wall with respect to the distance the bottom of the ladder is from the base of the wall after bottom has been pushed in three feet?



4) Water is being poured into a conical reservoir (Figure 4). The reservoir has a radius of 5 feet across the top and a height of 14 feet. At what rate is the volume V of the water changing with respect to the radius r when the depth h is 7 feet? (Hint: By similar triangles $5h = 14r$. See Figure 4)

The volume V of water in the reservoir is given by $V = \frac{1}{3}\pi r^2 h.$

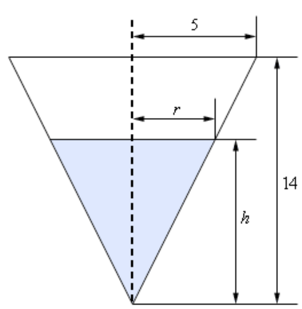
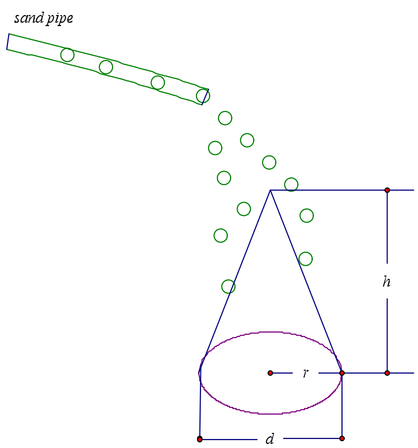


Figure 4

5) A 13-foot ladder is leaning against a vertical wall. If the bottom of the ladder is being pulled away from the wall at the rate of 2 feet per second, how fast is the area of the triangle formed by the wall, the ground and the ladder changing at the instant the bottom of the ladder is 12 feet from the wall?

6) Sand is pouring from a pipe at the rate of 16 cubic feet per second. If the falling sand forms a conical pile on the ground whose altitude h is always $1/4$ the diameter of the base d, how fast is the altitude increasing when the pile is 4 feet high?

Hint: See the Figure and use the fact that the volume $V = \frac{1}{3}\pi r^2 h$.



7) An object moves along the x axis. Its position at each $t \in [0,10]$ is given by $x(t) = t^3 - 9t^2 + 27t + 18$. Determine the

a. (2 pts) formula for the instantaneous velocity $v(t)$.

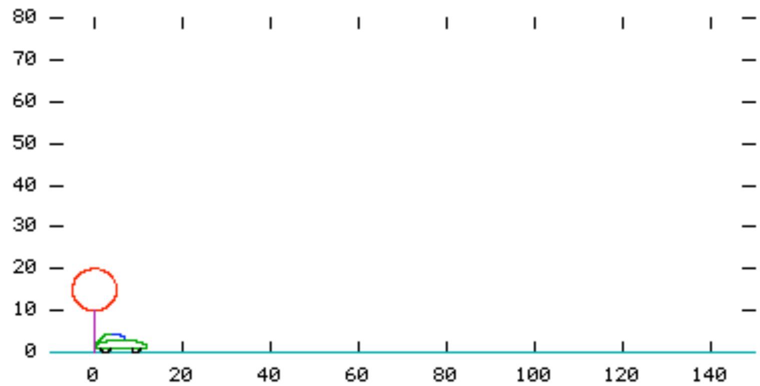
b. (2 pts) time(s) at which the object is changing direction, if ever.

c. (2 pts) formula for the instantaneous acceleration $a(t)$.

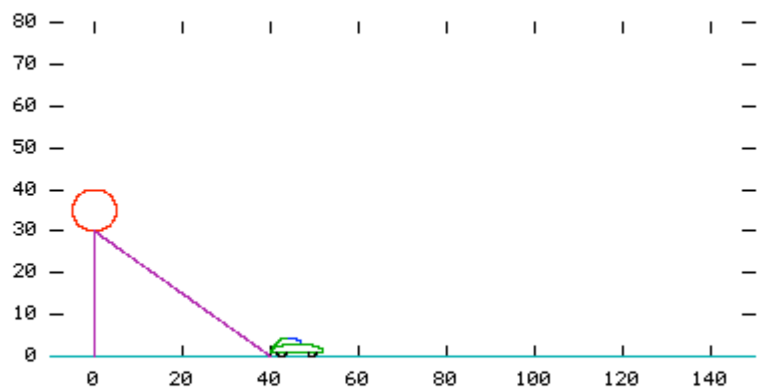
d. (6 pts) time interval(s), if any, during which the object is

i) speeding up: ______________________________? | ii) slowing down:_____________________________?

8) A large balloon is rising at the rate of 20 ft/sec. The balloon is 10 ft above the ground at the point in time that the back end of a car is directly below the bottom of the balloon (see first diagram). The car is traveling at 40 ft/sec. What is the rate of change of the distance between the bottom of the balloon and the point on the ground directly below the back of the car one second after the back of the car is directly below the balloon (see second diagram)?



Start



After 1 second

9) A circle starts out with a radius of 1 cm. (at time $t = 0$) and grows for $t \in [0, 40\pi]$. If the area of the circle is increasing at the rate of 2 cm^2 per second.

a. Find the rate of change of the radius with respect to time when the radius is 5 cm.

b. What is the radius when $t = 40\pi$?

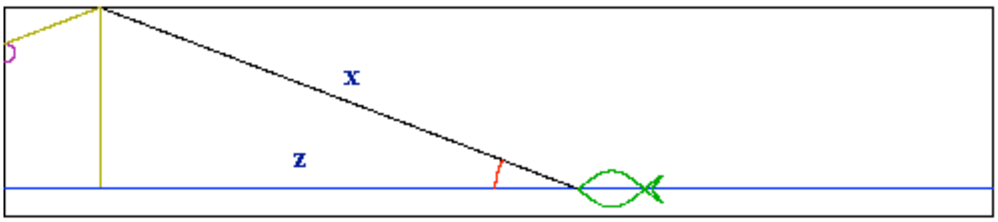
10) A sphere is increasing at a rate of 10 in^3/sec. Find the radius of this sphere at the moment its surface area is increasing at the rate of 5 in^2/sec.

11) A fish is reeled in at a rate of 1 foot per second from a point 10 feet above the water. At what rate is the angle between the fishing line and the water changing when there is a total of 25 feet of fishing line out?

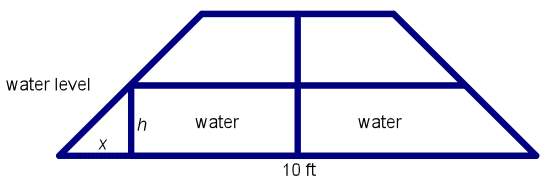
Hint: Let x = the amount of fishing line out.
Let z = the horizontal distance from the fish to the projection of the tip of the fishing pole onto the water.
Let θ = angle between the fishing line and the water.

Given: $\frac{dx}{dt} = -1\frac{ft}{\sec}$

Find: $\frac{d\theta}{dt}$



12) The dimensions of the cross sections of an isosceles trapezoidal tank consists of a base of 4 feet, a base of 10 feet, and a height of 12 feet, as shown in the figure below. It is also 100 feet wide (not shown). If the tank is filled by pumping water into it at a rate of 50 cubic feet per minute, how fast is the water level h rising when it is 4 feet deep? **Hint:** The volume of the isosceles trapezoid of water is $V = 100(10 - x)h$.



13) Two people are 50 feet apart. One of them starts walking north at a rate so that the angle shown in the diagram below is changing at a constant rate of 0.01 rad/min. At what rate is distance between the two people changing when $\theta = 0.5$ radians?

Mean Value Theorem

Inequality Preserving Limit Theorem

Let $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$.
If $f(x) \le g(x)$ for all $0 < \lvert x - c \rvert < p$, some $p > 0$, then $L \le M$.

Proof: By contradiction. Assume $L > M$. Let $\varepsilon = \dfrac{L - M}{2} > 0$. Then since $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$, there exists a $\delta > 0$ with $\delta < p$ such that, for $0 < \lvert x - c \rvert < \delta$,

a) $\underline{L - \varepsilon < f(x)} < L + \varepsilon$

b) $\underline{f(x) < g(x)}$

c) $M - \varepsilon < \underline{g(x) < M + \varepsilon}$

From a), b) and c), we obtain, for $0 < \lvert x - c \rvert < \delta$,

$$L - \varepsilon < f(x) \le g(x) < M + \varepsilon \Rightarrow L - \varepsilon < M + \varepsilon \Rightarrow L < M + 2\varepsilon = M + 2\frac{L - M}{2} = L \Rightarrow \underline{L < L}$$

This is a contradiction. □

Remark: This theorem is also true for one-sided limits.

Definition of Some Important Points

Let f be defined on an interval I that contains c.
c is an <u>end point</u> of $f \overset{def}{\equiv}$ if c is an end point of I. c is a <u>stationary point</u> of $f \overset{def}{\equiv}$ if (c is an interior point of I) $\wedge$ $f'(c) = 0$. c is a <u>singular point</u> of $f \overset{def}{\equiv}$ if (c is an interior point I) $\wedge$ $f'(c)$ is not defined (nd).

Definition of a Critical Point

Let f be defined on an interval I that contains c.
c is a <u>critical point</u> of $f \overset{def}{\equiv}$ if c is an end point, a singular point or a stationary point of f.

Find the critical points of $f(x) = \dfrac{\lvert x \rvert}{1 + x^2}$ on $[-2,2]$

Derivative of *f*	**Critical Points of *f***
$f'(x) = -\dfrac{x(x-1)(x+1)}{\lvert x \rvert (x^2+1)^2}$	$-2,2$ are end points 0 is a singular point $-1,1$ are stationary points

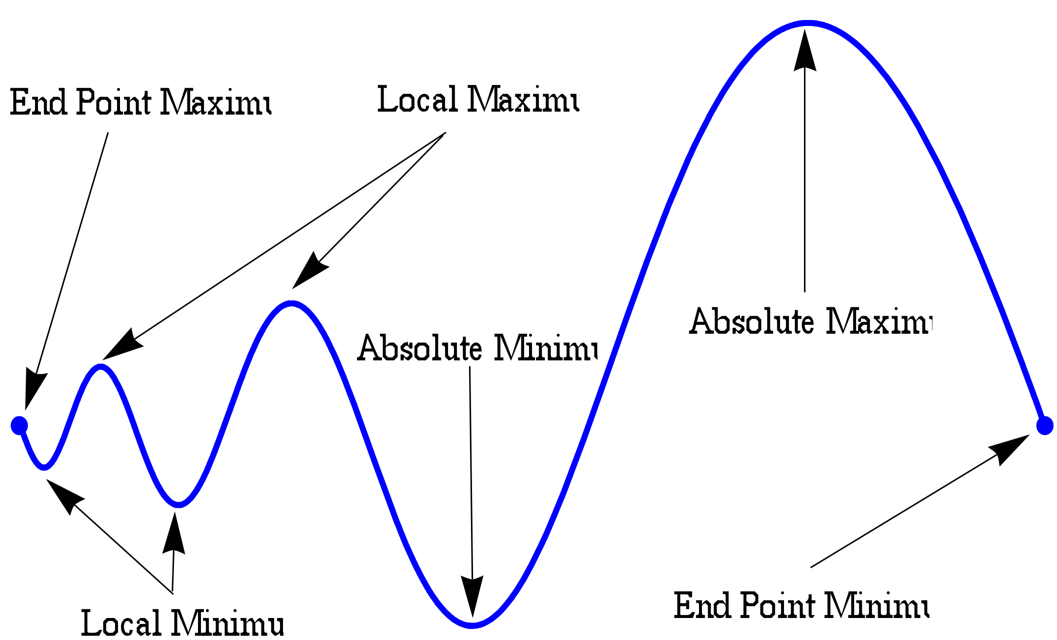
Let f be defined on an interval I that contains c. Then
f has an <u>absolute maximum/max</u> at $c \overset{\text{def}}{\equiv}$ if $f(c) \ge f(x),\ \forall x \in I$
f has an <u>absolute minimum/min</u> at $c \overset{\text{def}}{\equiv}$ if $f(c) \le f(x),\ \forall x \in I$

Definition of a Local Extremum

Let f be defined on an interval I that contains the interior point c. Then
f has a <u>local maximum/max</u> at $c \overset{\text{def}}{\equiv}$ if $\exists$ an open interval $J = (p - c, p + c) \subseteq I,\ p > 0, \ni$ $f(c) \ge f(x), \forall x \in J$.
f has a <u>local minimum/min</u> $c \overset{\text{def}}{\equiv}$ if $\exists$ an open interval $J = (p - c, p + c) \subseteq I,\ p > 0, \ni$ $f(c) \le f(x), \forall x \in J$.

Definition of an End Point Extremum

Let f be defined on an interval I that contains the end point c. Then
f has an <u>end point maximum/max</u> at $c \overset{\text{def}}{\equiv}$ if $\exists$ a half open subinterval $J = [c, p + c)$ or $J = (c - p, c]$ with $J \subseteq I,\ p > 0, \ni\ f(c) \ge f(x), \forall x \in J$.
f has an <u>end point minimum/min</u> at $c \overset{\text{def}}{\equiv}$ if $\exists$ a half open subinterval $J = [c, p + c)$ or $J = (c - p, c]$ with $J \subseteq I,\ p > 0, \ni\ f(c) \ge f(x), \forall x \in J$.



Critical Point Theorem (Fermat's Theorem)

Let f be defined on an interval I.
If f has an absolute, local or endpoint extremum at $c \in I$, then c is a critical point of f.

Proof: We may assume that c is neither an endpoint nor a singular point of f ; otherwise, we are done. Therefore c is an interior point of I and $f'(c)$ exists. Thus, f has a local extremum at c . We may also assume, without loss of generality, that f has a local maximum[3] at c . To finish, we must then show that $f'(c) = 0$. So, since f has a local maximum at c, $\exists$ an open interval J that contains c and is a subset of I such that $f(x) \le f(c)$, $\forall x \in J$, In particular, $\forall x \in J$,

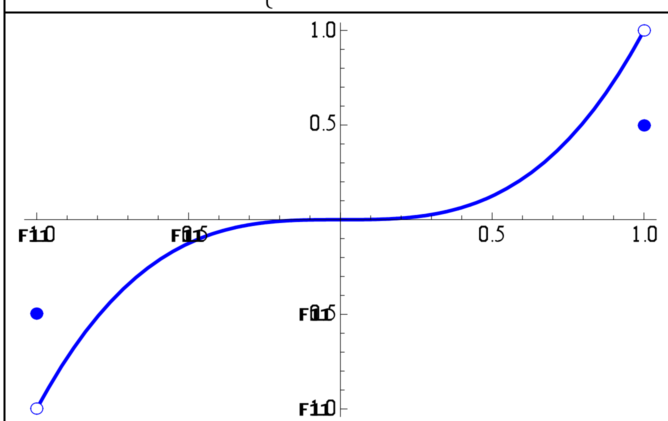
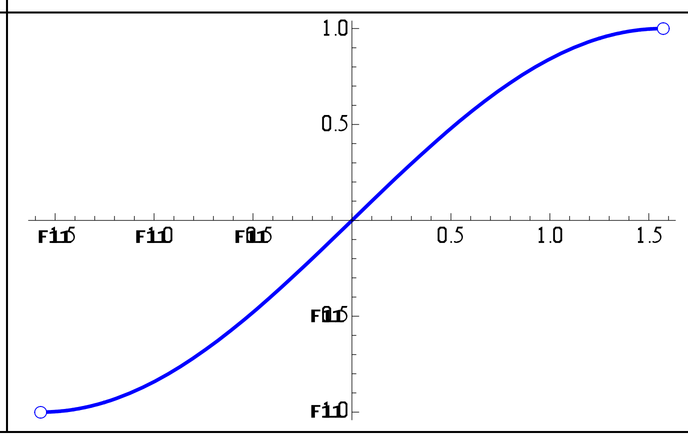
a) $x < c \Rightarrow < f(x) \le f(c) \Rightarrow \dfrac{f(x) - f(c)}{x - c} \ge 0 \Rightarrow f_-'(c) \overset{def}{\equiv} \lim_{x \to c^-} \dfrac{f(x) - f(c)}{x - c} \ge 0$

b) $x > c \Rightarrow f(x) \le f(c) \Rightarrow \dfrac{f(x) - f(c)}{x - c} \le 0 \Rightarrow f_+'(c) \overset{def}{\equiv} \lim_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \le 0$.

But, since $f'(c)$ exists, $f_-'(c) = f'(c) = f_+'(c)$. Thus, from a) and b), we then obtain that $f'(c) \ge 0$ and $f'(c) \le 0$. Hence, $f'(c) = 0$.

Max-Min Existence Theorem (MMT): If f is continuous on the closed interval $[a, b]$, then f attains an absolute maximum and minimum in $[a, b]$.

Remark: If either the 'continuous' requirement or the 'closed interval' requirement is dropped from the Max-Min Existence Theorem, then the theorem is false. For example,

Continuity Dropped	**Closed Interval Dropped**
$f(x) = \begin{cases} -0.5, & \text{if } x = -1 \\ x^3, & \text{if } -1 < x < 1 \\ 0.5, & \text{if } x = 1 \end{cases}$	$f(x) = \sin x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
	

[3] Similar proof if f has a local minimum at c .

Example:

Find the absolute maximum and minimum of $f(x)=\frac{|x|}{1+x^2}$ on $[-2,2]$.

The Max-Min Theorem guarantees that f attains both a maximum and minimum on $[-2,2]$ and the Critical Point Theorem guarantees that these extrema occur at critical points of f.	
Critical Points	$f(c)$
$-2,2$ are end points 0 is a singular point $-1,1$ are stationary points	$f(-2)=f(2)=0.4$ $f(0)=0 \leftarrow$ absolute minimum $f(-1)=f(1)=0.5 \leftarrow$ absolute maximum

Exercises:

1) Let $f(x)=\frac{x}{x^2+4}$, $x\in[-1,3]$.
   a) State the Max-Min Existence Theorem.
   b) State the Critical Point Theorem.
   c) Determine the critical points of $f(x)$.
   d) Determine the absolute maximum and minimum of $f(x)$.

Find the absolute maximum and minimum values of f in problems 3)-7).

2) $f(x)=x^3-3x^2+1$ on $\left[-\frac{1}{2},4\right]$.

3) $f(x)=\frac{x}{x^2+1}$ on $[0,2]$

4) $f(x)=x-\ln x$ on $\left[\frac{1}{2},2\right]$.

5) $f(x)=xe^{-x^2/8}$ on $[-1,4]$

6) $f(x)=x^a(1-x)^b$ on $[0,1]$.

7) $f(x)=x^{3/5}(4-x)$ on $[0,5]$.

8) A model of the velocity of the space shuttle Discovery that deployed the Hubble Space Telescope in 1990 from liftoff at $t=0$ until the solid rocket boosters were jettisoned at $t=126$ seconds is given by $v(t)=0.001302t^3-0.09029t^2+23.61t-3.083$. Find the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

Rolle's Theorem (RT)

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b).
If $f(a)=f(b)=0$, then $\exists c\in(a,b)\ni f'(c)=0$.

Proof: The theorem is trivial if f is constant on $[a,b]$. Therefore, we may assume that f is not constant on $[a,b]$. By the Max-Min Theorem, f has both an absolute maximum and absolute minimum on $[a,b]$. Clearly, both the maximum and minimum cannot occur at the end points of $[a,b]$; otherwise, f would be constant on $[a,b]$. We may therefore assume that f has an extremum at some $c\in(a,b)$. By the Critical Point Theorem, since c is neither a singular point nor an end point of f on $[a,b]$, we conclude that $f'(c)=0$. ∎

Mean Value Theorem (MVT)

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then
$\exists\, c\in(a,b)\ni f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Proof: Define $F(x)=f(x)-l(x)$, where $l(x)=\dfrac{f(b)-f(a)}{b-a}(x-a)+f(a)$, the line determined by the points $(a,f(a))$ and $(b,f(b))$. Now apply Rolle's Theorem to $F(x)$ on $[a,b]$. Therefore, $\exists\, c\in(a,b)\ni\ F'(c)=0$. That is,
$0=F'(c)=f'(c)-l'(c)=f'(c)-\dfrac{f(b)-f(a)}{b-a}\Rightarrow\ f'(c)=\dfrac{f(b)-f(a)}{b-a}$. ∎

Examples:

1) Prove that the equation $x^3+x+1=0$ has exactly one real root.

   Proof: Let $f(x)=x^3+x+1$. We will first show that f has at least one zero: Applying the Intermediate Value Theorem to f on $[-1,1]$, we see that f is continuous on $[-1,1]$, and, $f(-1)<0$ and $f(1)>0$. Thus, f has at least one real zero in $[-1,1]$. We now show this is the only real zero f of on $[-1,1]$. By contradiction, assume that f has at least two zeros in $(-1,1)$, say, $f(a)=0$ and $f(b)=0$, with $a<b$ and $a,b\in(-1,1)$. Then, by Rolle's Theorem, $\exists c\in(a,b)\subseteq(-1,1)\ni f'(c)=0$. But $f'(x)=3x^2+1>0$ on $(-1,1)$. This is a contradiction . ∎

2) Suppose an object moves in a straight line with position function $s=f(t)$. If $s=f(t)$ is continuous on $[a,b]$ and differentiable on (a,b), prove its average velocity from $t=a$ to $t=b$ is taken on as an instantaneous velocity by our object for some $t=c\in(a,b)$.

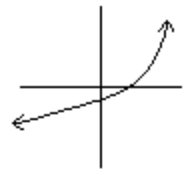
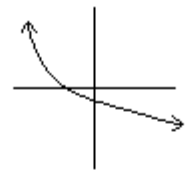
   Proof: The average velocity of $s=f(t)$ over $[a,b]$ is given by $\dfrac{f(b)-f(a)}{b-a}$. Thus, by the Mean Value Theorem, $\exists\, c\in(a,b)\ni\dot{s}(t)=f'(c)=\dfrac{f(b)-f(a)}{b-a}$. ∎

3) If f is continuous on $[0,2]$ and differentiable on $(0,2)$, with $f(0)=-3$ and $f'(x)\le 5$ for all values of $x\in(0,2)$, how large can $f(2)$ possibly be?

   Proof: By the Mean Value Theorem, we have $f'(c)=\dfrac{f(2)-f(0)}{2-0}$, for some $c\in(0,2)$. This implies that $f(2)=2f'(c)+f(0)\le 2\cdot 5-3=7$. Thus, $f(2)\le 7$. ∎

Monotonicity

Definition of a Strictly Monotonic Function

Let f be defined on an interval I. Let $x_1 \in I$ and $x_2 \in I$.	
f is <u>increasing</u> $(f \uparrow)$ on $I \overset{def}{\equiv}$ if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$	
f is <u>decreasing</u> $(f \downarrow)$ on $I \overset{def}{\equiv}$ if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$	

Monotonicity Theorem

Let f be continuous on an interval I and differentiable on the interior of I.	
(i)	$f'(x) > 0$ on the interior of $I \Rightarrow f \uparrow$ on all of I
(ii)	$f'(x) < 0$ on the interior of $I \Rightarrow f \downarrow$ on all of I
(iii)	$f' \equiv 0$ on the interior of $I \Rightarrow f$ is constant on all of I

Proof of (i): Let $x_1, x_2 \in I$ with $x_1 < x_2$. To show $f(x_1) < f(x_2)$. Apply the MVT to f on $[x_1, x_2]$. Therefore, there exists some $c \in (x_1, x_2)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $x_2 - x_1 > 0$ and $f'(c) > 0$, we have that $f(x_2) - f(x_1) > 0$ or $f(x_1) < f(x_2)$. Thus, $f \uparrow$ on all of I. ∎

Proof of (ii): Let $x_1, x_2 \in I$ with $x_1 < x_2$. To show $f(x_1) > f(x_2)$. Apply the MVT to f on $[x_1, x_2]$. Therefore, there exists some $c \in (x_1, x_2)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $x_2 - x_1 > 0$ and $f'(c) < 0$, we have that $f(x_2) - f(x_1) < 0$ or $f(x_1) > f(x_2)$. Thus, $f \downarrow$ on all of I. ∎

Proof of (iii): Let $x_1, x_2 \in I$ with $x_1 \neq x_2$. To show $f(x_1) = f(x_2)$. May assume that $x_1 < x_2$. Apply the MVT to f on $[x_1, x_2]$. Therefore, there exists some $c \in (x_1, x_2)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $x_2 - x_1 > 0$ and $f'(c) = 0$, we have that $f(x_2) - f(x_1) = 0$ or $f(x_1) = f(x_2)$. Thus, since $x_1, x_2 \in I$ were arbitrary, f is constant on all of I. ∎

Corollaries

Let f, g be continuous on an interval I and differentiable on the interior of I. Then	
(i)	$f' \equiv 0$ on the interior of $I \Leftrightarrow f$ is constant on all of I
(ii)	$f' \equiv g'$ on the interior of $I \Leftrightarrow f, g$ differ by some constant on I

Monotonic Endpoint Extension Theorem

Let f be continuous at a and b, and monotonic on the open interval (a,b). Then
f is monotonic on the closed interval $[a,b]$.

Proof: Without loss of generality, we may assume that $f \uparrow$ on (a,b) (if $f \downarrow$ on (a,b), consider $-f$). Let $x \in (a,b)$. To show that $f(a) < f(x) < f(b)$. Choose $x_1, x_2 \in (a,b)$ with $x_1 < x < x_2$. Let $\delta = \min(x_1 - a, b - x_2)$. Then

$$0 < \Delta x < \delta \Rightarrow a + \Delta x < x_1 < x < x_2 < b - \Delta x$$

Since $f \uparrow$ on (a,b), we have, for $0 < \Delta x < \delta$,

$$f(a+\Delta x) < f(x_1) < f(x) < f(x_2) < f(b - \Delta x)$$

By the Inequality Preserving Limit Theorem and the continuity of f at a and b, we have

(*) $$f(a) = \lim_{\Delta x \to 0^+} f(a+\Delta x) \le \lim_{\Delta x \to 0^+} f(x_1) \le \lim_{\Delta x \to 0^+} f(x) \le \lim_{\Delta x \to 0^+} f(x_2) \le \lim_{\Delta x \to 0^+} f(b-\Delta x) = f(b)$$

Since $\lim_{\Delta x \to 0^+} f(x_1) = f(x_1)$, $\lim_{\Delta x \to 0^+} f(x) = f(x)$, and $\lim_{\Delta x \to 0^+} f(x_2) = f(x_2)$, we have from (*)

$$f(a) =\le f(x_1) < f(x) < f(x_2) \le f(b)$$

Thus, $f(a) =< f(x) < f(b)$. □

Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ ↑ and where $f \downarrow$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2)$	
We use the Monotonicity Theorem to construct the Sign Chart of f'	
Thus $f \downarrow$ on $(-\infty,-1] \cup [0,2]$ and $f \uparrow$ $[-1,0] \cup [2,\infty)$	Sign Chart of f' 0 0 0 ← − − − − (−1) + + + + 0 − − − − (2) + + + + →

Find where $f(x) = \dfrac{x^2 - 3}{x^3}$ ↑ and where $f \downarrow$

$f'(x) = \dfrac{-(x^2-9)}{x^4} = -\dfrac{(x-3)(x+3)}{x^4}$
We use the Monotonicity Theorem to construct the Sign Chart of f'
Sign Chart of f' 0 ND 0 ← − − − − − − − − − (−3) + + + + + + + + 0 + + + + + + + + (3) − − − − − − − − − →
The Sign Chart tells us that $f \downarrow (-\infty,-3] \cup [3,\infty)$ and $f \uparrow$ on $[-3,0) \cup (0,3]$

Find where $f(x)=\frac{x}{1+x^2}\uparrow$ **and where** $f\downarrow$

$f'(x)=\frac{1-x^2}{(1+x^2)^2}=-\frac{(x-1)(x+1)}{(1+x^2)^2}$
We use the Monotonicity Theorem to construct the Sign Chart of f'
Sign Chart of f' $\leftarrow ---------- \overset{0}{(-1)} ++++++++0+++++++ \overset{0}{(+1)} ---------- \rightarrow$
The Sign Chart tells us that $f\downarrow (-\infty,-1]\cup[1,\infty)$ and $f\uparrow$ on $[-1,1]$

Find where $f(x)=4x^3-3x^2-6x+12\uparrow$ **and where** $f\downarrow$

$f'(x)=12x^2-6x-6=6(2x^2-x-1)=6(2x+1)(x-1)$
We use the Monotonicity Theorem to construct the Sign Chart of f'
Sign Chart of f' $\leftarrow +++++++++ \overset{0}{\left(-\frac{1}{2}\right)} ----0-------- \overset{0}{(1)} +++++++++ \rightarrow$
The Sign Chart tells us that $f\uparrow$ on $(-\infty,-1/2]\cup[1,\infty)$ and $f\downarrow [-1/2,1]$

Find where $f(x)=400\frac{x+2}{x^3}\uparrow$ **and where** $f\downarrow$

$f'(x)=-800\frac{(x+3)}{x^4}$
We use the Monotonicity Theorem to construct the Sign Chart of f'
Sign Chart of f' $\leftarrow +++++++++++++ \overset{0}{(-3)} ------ \overset{ND}{0} ------------- \rightarrow$
The Sign Chart tells us that $f\uparrow (-\infty,-3]$ and $f\downarrow$ on $[-3,0)\cup(0,\infty)$

Speeding Up/Slowing Down Theorem

Consider some object in rectilinear motion on the real line with position $x(t)$ and velocity $\dot{x}(t) = v(t)$ during the interval of time $[\alpha, \beta]$. Let $v(t)$ be continuous on $[\alpha, \beta]$ and differentiable on (α, β). Then

1. If $\dot{x}(t) = v(t) < 0$ and $a(t) = \dot{v}(t) < 0$ on (α, β), then

   i) The object is Speeding Up $(|v(t)| \uparrow)$ on $[\alpha, \beta]$.

   ii) The object is moving Toward the Left $(v(t) \le 0)$ on $[\alpha, \beta]$.

1. If $\dot{x}(t) = v(t) > 0$ and $a(t) = \dot{v}(t) > 0$ on (α, β), then

   i) The object is Speeding Up $(|v(t)| \uparrow)$ on $[\alpha, \beta]$.

   ii) The object is moving Toward the Right $(v(t) \ge 0)$ on $[\alpha, \beta]$.

2. If $\dot{x}(t) = v(t) > 0$ on (α, β) and $a(t) = \dot{v}(t) < 0$ on (α, β), then

   i) The object is Slowing Down $(|v(t)| \downarrow)$ on $[\alpha, \beta]$.

   ii) The object is moving Toward the Right $(v(t) \ge 0)$ on $[\alpha, \beta]$.

3. If $\dot{x}(t) = v(t) < 0$ on (α, β) and $a(t) = \dot{v}(t) > 0$ on (α, β), then

   i) The object is Slowing Down $(|v(t)| \downarrow)$ on $[\alpha, \beta]$.

   ii) The object is moving Toward the Left $(v(t) \le 0)$ on $[\alpha, \beta]$.

Proof: We shall only prove case 1, since the other cases have a similar proof. Now since $a(t) = \dot{v}(t) < 0$ on (α, β), by the Monotonicity Theorem, $v(t) \downarrow$ $[\alpha, \beta]$. Thus, $|v(t)| \uparrow$ on $[\alpha, \beta]$. This says our object is speeding up on $[\alpha, \beta]$. Again, by the Monotonicity Theorem, since $\dot{x}(t) = v(t) < 0$ on (α, β), $x(t) \downarrow$ on $[\alpha, \beta]$. This says our object is moving toward the left on $[\alpha, \beta]$.

Concavity

Definition of Concavity

Let *f* be differentiable on an open interval I
f (as well as its graph) is <u>concave up</u> on $I \overset{def}{\equiv}$ if $f' \uparrow$ on I. f (as well as its graph) is <u>concave down</u> on $I \overset{def}{\equiv}$ if $f' \downarrow$ on I.

Concavity Theorem

Let *f* be twice differentiable on an open interval *I*.
$f''(x) > 0$ on $I \Rightarrow f' \uparrow$ on $I \Leftrightarrow f$ is concave up on I. $f''(x) < 0$ on $I \Rightarrow f' \downarrow$ on $I \Leftrightarrow f$ is concave down on I

Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ is concave up and where it is concave down.

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2)$ $f''(x) = 36x^2 - 24x - 24 = 36\left(x - \left\{\frac{1-\sqrt{7}}{3}\right\}\right)\left(x - \left\{\frac{1+\sqrt{7}}{3}\right\}\right)$
We use the Concavity Theorem to construct the Sign Chart of f''
Sign Chart of f'' $\cup \qquad \cap \qquad \cup$ $\leftarrow + + + + + + + + + \left(\frac{1-\sqrt{7}}{3}\right) - - 0 - - - - - - - - \left(\frac{1+\sqrt{7}}{3}\right) + + + + + + + + + \rightarrow$
f is concave up on $\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$ f is concave down on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$ f is concave up on $\left(\frac{1+\sqrt{7}}{3}, \infty\right)$

Definition of an Inflection Point

Let f be continuous at c.
$(c, f(c))$ is an <u>inflection point</u> of $f \overset{def}{\equiv}$ if $\exists$ open intervals $(a,c),(c,b) \ni f' \uparrow (\downarrow)$ on $(a,c) \wedge f' \downarrow (\uparrow)$ on (c,b).

Critical Point Theorem for Inflection Points

Let $(c, f(c))$ be an inflection point of f. Then
c is a critical point of f', that is, $f''(c) = 0$ or $f''(c)$ is not defined.

Proof: Assume it is not the case that $f''(c)$ is not defined. Therefore $f''(c)$ exists and, hence, f' is continuous at c. Now since $(c, f(c))$ is an inflection point of f, $\exists$ open intervals $(a,c),(c,b) \ni f' \uparrow (\downarrow)$ on $(a,c) \wedge f' \downarrow (\uparrow)$ on (c,b). We may assume without loss of generality that $f' \uparrow$ on $(a,c) \wedge f' \downarrow$ on (c,b)[4]. Therefore $f' \uparrow$ on $(\mathrm{a,c}] \wedge f' \downarrow$ on $[\mathrm{c,b})$ because f' is continuous at c. Thus $f'(x) \le f'(c), \forall x \in (a,b)$. Since $f''(c)$ exists, this implies that $f''(c) = 0$, as in the proof of the Critical Point Theorem.

Find the Inflection Points of $f(x) = 4x^3 - 3x^2 - 6x + 12$

$f'(x) = 12x^2 - 6x - 6 = 6(2x^2 - x - 1)$	
$f''(x) = 6(4x - 1)$	Inflection Points Possibly Occur at
	$1/4$
We use the Concavity Theorem to construct the Sign Chart of f''	
Sign Chart of $f''(x)$ $\cap$ 0 $\cup$ $\leftarrow$ − − − − − − − − − 0 − − − $(1/4)$ + + + + + + + + + + + + + + + + ++ $\rightarrow$	
The Sign Chart tells us that an inflection point occurs at $\frac{1}{4}$	

Find the Inflection Points of $f(x) = \dfrac{x}{1 + x^2}$

$f'(x) = \dfrac{1 - x^2}{(1 + x^2)^2}$	
$f''(x) = \dfrac{2x(x - \sqrt{3})(x + \sqrt{3})}{(1 + x^2)^3}$	Inflection Points possibly occur at
	$0, -\sqrt{3}$, or $\sqrt{3}$
We use the Concavity Theorem to construct the Sign Chart of f''	
Sign Chart of $f''(x)$ $\cap$ 0 $\cup$ $\cap$ 0 $\cup$ $\leftarrow$ − − − − − − − − − − − $(-\sqrt{3})$ + + + + + 0 − − − − − $(\sqrt{3})$ + + + + + + + + + ++ $\rightarrow$	
The Sign Chart tells us that inflection points occur at $0, -\sqrt{3}$, and $\sqrt{3}$.	

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[4] Similar proof if $f' \downarrow$ on $(a,c) \wedge f' \uparrow$ on (c,b).

Find the Inflection Points of $f(x) = 400\dfrac{x+2}{x^3}$

$f'(x) = -800\dfrac{(x+3)}{x^4}$	
$f''(x) = 2400\dfrac{(x+4)}{x^5}$	Inflection Points possibly occur at
	-4
We use the Concavity Theorem to construct the Sign Chart of f''	
Sign Chart of $f''(x)$ $\cup$ 0 $\cap$ ND $\cup$ $\leftarrow + + + + + + + + + + (-4) - - - - - - - - - - - - 0 + + + + + + + + + \rightarrow$	
The Sign Chart tells us that an inflection point occurs at -4	

Sign Preserving Limit Theorem

Let $\lim\limits_{x \to c} f(x) = L \neq 0$. Then
$\exists \delta > 0$ such that $x \in (c-\delta, c) \cup (c, c+\delta) \Rightarrow f(x)$ keeps the same sign as L.

Proof: Let $\varepsilon = \dfrac{|L|}{2} > 0$. Then, since $\lim\limits_{x \to c} f(x) = L$, there exists a $\delta > 0$ such that

$$x \in (c-\delta, c) \cup (c, c+\delta) \Leftrightarrow 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon \Leftrightarrow L - \varepsilon < f(x) < L + \varepsilon .$$

Case $L > 0$: $L - \varepsilon < f(x) < L + \varepsilon \Leftrightarrow \underline{L - \dfrac{L}{2}} < f(x) < L + \dfrac{L}{2} \Rightarrow \underline{0 < \dfrac{L}{2} < f(x)} \Rightarrow f(x) > 0$

Case $L < 0$: $L - \varepsilon < f(x) < L + \varepsilon \Leftrightarrow L + \dfrac{L}{2} < f(x) < \underline{L - \dfrac{L}{2}} \Rightarrow \underline{f(x) < \dfrac{L}{2} < 0} \Rightarrow f(x) < 0$. □

Remark: This theorem is also true for one-sided limits.

First Derivative Test Theorem

	Let f be continuous on an open interval (a,b) that contains the critical point c. Then
(i)	$f' > 0$ on (a,c) and $f' < 0$ on $(c,b) \Rightarrow f(c)$ is a local maximum.
(ii)	$f' < 0$ on (a,c) and $f' > 0$ on $(c,b) \Rightarrow f(c)$ is a local minimum.
(iii)	f' keeps the same sign on both sides of $c \Rightarrow f(c)$ is not a local extreme value.

Proof of (i): Assume $f' > 0$ on (a,c) and $f' < 0$ on (c,b). Since f is continuous on both $(a,c]$ and $[c,b)$, we have by the Monotonicity Theorem that $f \uparrow$ on $(a,c]$ and $f \downarrow [c,b)$. Thus, $f(c)$ is a local maximum on (a,b). □

Proof of (ii): Assume $f' < 0$ on (a,c) and $f' > 0$ on (c,b). Since f is continuous on both $(a,c]$ and $[c,b)$, we have by the Monotonicity Theorem that $f \downarrow$ on $(a,c]$ and $f \uparrow [c,b)$. Thus, $f(c)$ is a local minimum on (a,b). □

Proof of (iii): Similar to the proofs of (i) and (ii), in which we either have $f \uparrow$ on (a,b) or $f \downarrow$ on (a,b). Thus $f(c)$ can not be a local extreme value. ▯

Second Derivative Test Theorem

Suppose $f'(c)=0$ and $f''(c)$ exists. Then	
(i)	$f''(c)>0 \Rightarrow f(c)$ is a local minimum.
(ii)	$f''(c)<0 \Rightarrow f(c)$ is a local maximum.

Proof of (i): Assume $f''(c)>0$. So f' is continuous at c. Since $f''(c)=\lim_{x\to c}\frac{f'(x)-f'(c)}{x-c}=\lim_{x\to c}\frac{f'(x)}{x-c}>0$, $\exists\delta>0$ such that $\frac{f'(x)}{x-c}>0$, when $0<|x-c|<\delta$ or $x\in(c-\delta,c)\cup(c,c+\delta)$. Thus,

$$\left.\begin{cases} c-\delta<x<c \Rightarrow x-c<0 \Rightarrow \underline{f'<0 \text{ on } (c-\delta,c)} \\ c<x<c+\delta \Rightarrow x-c>0 \Rightarrow \underline{f'>0 \text{ on } (c,c+\delta)} \end{cases}\right\} \Rightarrow \text{by the First Derivative Test, } f(c) \text{ is a local minimum.}$$

Proof of (ii): Assume $f''(c)<0$. So f' is continuous at c. Since $f''(c)=\lim_{x\to c}\frac{f'(x)-f'(c)}{x-c}=\lim_{x\to c}\frac{f'(x)}{x-c}<0$, $\exists\delta>0$ such that $\frac{f'(x)}{x-c}<0$, when $0<|x-c|<\delta$ or $x\in(c-\delta,c)\cup(c,c+\delta)$. Thus,

$$\left.\begin{cases} c-\delta<x<c \Rightarrow x-c<0 \Rightarrow \underline{f'>0 \text{ on } (c-\delta,c)} \\ c<x<c+\delta \Rightarrow x-c>0 \Rightarrow \underline{f'<0 \text{ on } (c,c+\delta)} \end{cases}\right\} \Rightarrow \text{by the First Derivative Test, } f(c) \text{ is a local maximum.}$$ ▯

Exercises:

1. One end of a 27-foot ladder rests on the ground and the other end rests on the top of an 8-foot wall. As the bottom of the ladder is pushed along the ground toward the wall, the top extends beyond the wall. Find the maximum *horizontal* overhang of the top end (see Figure 6).

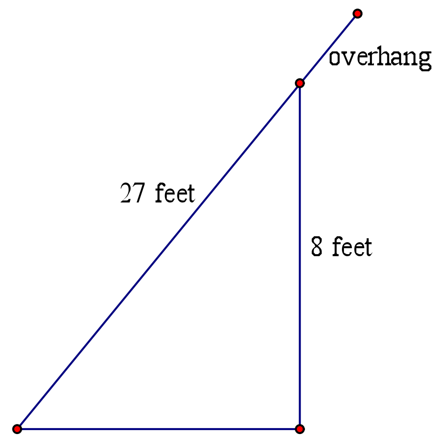
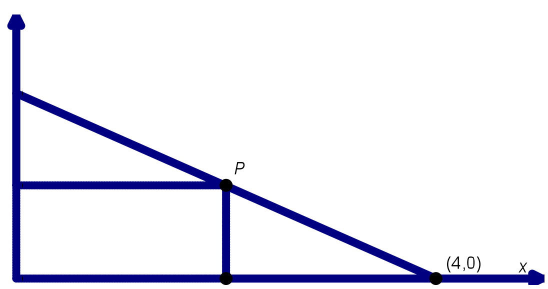
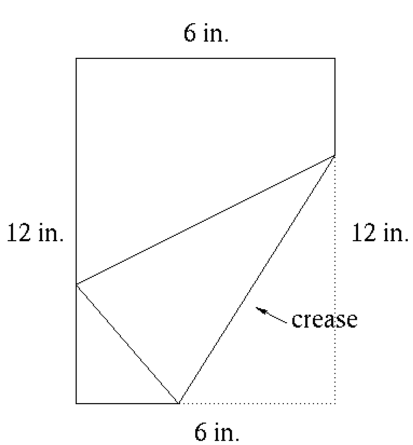


Figure 6

2) Find the coordinates of *P* that maximizes the area of the rectangle shown in the figure below.



3) A rectangular piece of paper is 12 inches high and six inches wide. The lower right-hand corner is folded over so as to reach the leftmost edge of the paper (See figure.). Find the minimum length of the resulting crease.



4) Find the point (x, y) on the graph of $y = \sqrt{x}$ that is nearest to the point $(4,0)$ by determining the following, where S denotes the function that is to be minimized.

e) $S(x) =$ $D_s =$

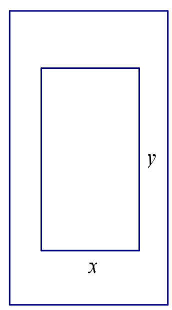
f) $S'(x) =$

g) Critical points of $S(x) = \{ \quad \}$

h) Test the Critical Points:

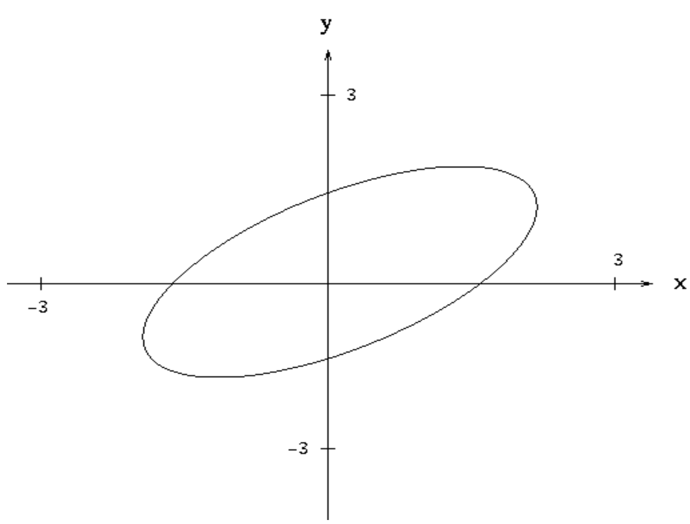
i) Answer: the point (x, y) on the graph of $y = \sqrt{x}$ that is nearest to the point $(4,0) = (\quad , \quad)$

5) An advertising flyer is to contain 50 square inches of printed matter, with 2-inch margins at the top and bottom and 1-inch margins on each side. What dimensions for the flyer would use the least paper?

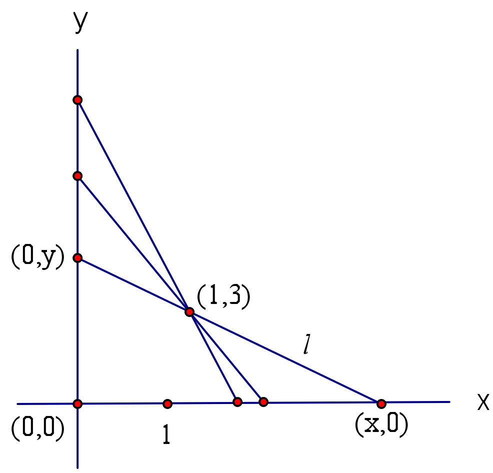


6) An advertising flyer is to have an area of 50 square inches, in which the printed matter will have 2-inch margins at the top and bottom and 1-inch margins on each side. What dimensions for the flyer would maximize the printed matter?

7) The graph of $x^2 - xy + y^2 = 3$ is a "tilted" ellipse (See diagram.). Among all points (x, y) on this graph, find the largest and smallest values of y. Among all points (x, y) on this graph, find the largest and smallest values of x.

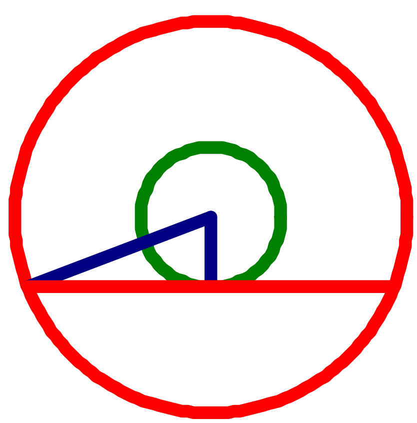


8) Problem: There are infinitely many lines that pass through the point $(1,3)$ and form a right triangle with the coordinate axes (see the accompanying figure). In this problem, you are to justify the existence of that line through $(1,3)$ that minimizes the area of the right triangle thus formed by answering the following questions:



a) State the Max-Min Theorem (MMT).
b) Determine a function $A(x)$ that expresses the area of any of the right triangles formed by the axes and a line that passes through $(1,3)$.
c) Determine the domain Dom(A)
d) Apply MMT to $A(x)$

9) A humidifier uses a rotating disk of radius r, which is partially submerged in water (shown in the lower region of the figure labeled WATER). The most evaporation occurs when the exposed wetted region (shown as the upper region of the figure labeled WETTED) is maximized. Show this happens when h (the displacement from the center to the water) is equal to $r/\sqrt{1+\pi^2}$.



10) The function $f(x)=\dfrac{Ax}{x^2+B^2}$ has a local minimum at $x=-2$ and satisfies $f'(0)=1$. Find A and B.

11) What are the dimensions and the volume of the square pyramid that can be cut and folded from a square piece of cardboard 20 by 20 square inches as illustrated in Figure 4 so that its volume is a maximum?

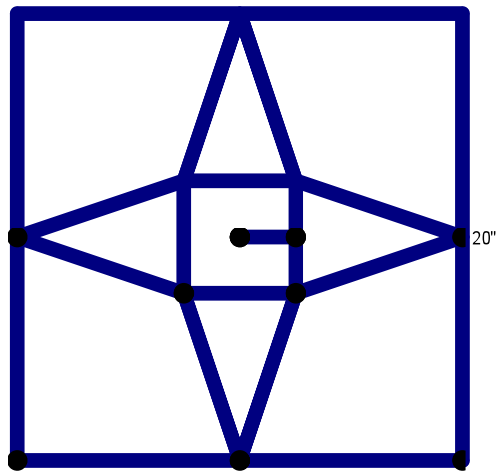


Figure 4

12) What are the dimensions and the volume of the square pyramid that can be cut and folded from a square piece of cardboard 20 by 20 square inches as illustrated in Figure 5 so that its volume is a maximum?

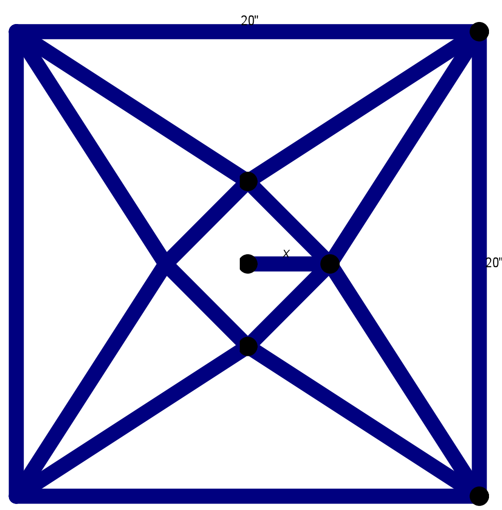


Figure 5

13) Which of the two constructed square pyramids from the above problems has the larger volume?

Asymptotes

Definition of a Horizontal Asymptote (H.A.)

The line $y = b$ is a <u>horizontal asymptote</u> to the graph of $f \overset{def}{\equiv}$ if $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

Find the horizontal asymptotes of $f(x) = \frac{x+2}{x^3}$

Sign Chart of f 0 ND $\leftarrow + + + + + + + + + + + + + (-2) - - - - 0 + + + + + + + + + + + \rightarrow$
The line $y = 0$ is an H.A. $\lim_{x\to-\infty} \frac{x+2}{x^3} = 0^+$ and $\lim_{x\to+\infty} \frac{x+2}{x^3} = 0^+$

Definition of a Vertical Asymptote (V.A.)

The line $x = a$ is a <u>vertical asymptote</u> to the graph of $f \overset{def}{\equiv}$ if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$.

Find the vertical asymptotes of $f(x) = \frac{x+2}{x^3}$

Sign Chart of f 0 ND $\leftarrow + + + + + + + + + + + + + (-2) - - - - 0 + + + + + + + + + + + \rightarrow$
The line $x = 0$ is a V.A. $\lim_{x\to 0^-} \frac{x+2}{x^3} = -\infty$ and $\lim_{x\to 0^+} \frac{x+2}{x^3} = \infty$

Definition of an Oblique Asymptote (O.A.)

The line $y = ax + b$ is an <u>oblique asymptote</u> to the graph of $f \overset{def}{\equiv}$ if $\lim_{x\to\infty}[f(x) - (ax+b)] = 0$ or $\lim_{x\to-\infty}[f(x) - (ax+b)] = 0$.

Find the oblique asymptotes of $f(x)=\dfrac{x^2+x+1}{x+1}$

The line $y=x$ is an O.A.
$f(x)=\dfrac{x^2+x+1}{x+1}=x+\dfrac{1}{x+1}\Rightarrow \lim\limits_{x\to\pm\infty}[f(x)-x]=\lim\limits_{x\to\pm\infty}\dfrac{1}{x+1}=0$

Find the oblique asymptotes of $f(x)=\dfrac{2x^3+x^2+2x+1}{x^2+1}$

The line $y=2x+1$ is an O.A.
$f(x)=\dfrac{2x^3+x^2+2x+1}{x^2+1}=2x+1+\dfrac{2}{x^2+1}\Rightarrow \lim\limits_{x\to\pm\infty}[f(x)-(2x+1)]=\lim\limits_{x\to\pm\infty}\dfrac{2}{x^2+1}=0$

Find the horizontal asymptotes of $f(x)=\dfrac{x}{1+x^2}$

Sign Chart of f
$\leftarrow ------------------0 +++++++++++++ \rightarrow$
The line $y=0$ is an H.A.
$\lim\limits_{x\to-\infty}\dfrac{x}{1+x^2}=0^-$ and $\lim\limits_{x\to\infty}\dfrac{x}{1+x^2}=0^+$

Find the horizontal asymptotes of $f(x)=400\dfrac{x+2}{x^3}$

Sign Chart of f
0 ND
$\leftarrow +++++++++++ (-2) ------ 0 ++++++++++++ \rightarrow$
The line $y=0$ is an H.A.
$\lim\limits_{x\to-\infty}400\dfrac{x+2}{x^3}=0^+$ and $\lim\limits_{x\to+\infty}400\dfrac{x+2}{x^3}=0^+$

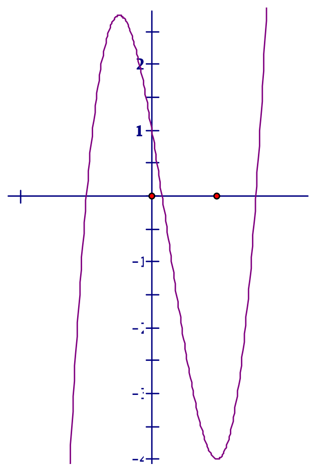
Find the vertical asymptotes of $f(x)=400\dfrac{x+2}{x^3}$

Sign Chart of f
0 ND
$\leftarrow +++++++++++ (-2) ------ 0 ++++++++++++ \rightarrow$
The line $x=0$ is a V.A.
$\lim\limits_{x\to0^-}400\dfrac{x+2}{x^3}=-\infty$ and $\lim\limits_{x\to0^+}400\dfrac{x+2}{x^3}=\infty$

Curve Sketching

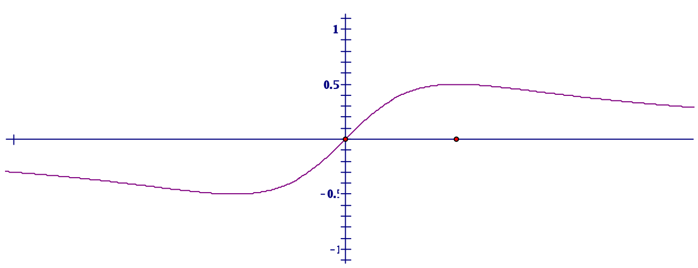
Sketch the graph of $f(x) = 4x^3 - 3x^2 - 6x + 1$

$f'(x) = 6(2x^2 - x - 1) = 6(2x+1)(x-1)$ $\qquad$ $f''(x) = 6(4x-1)$
$f\uparrow$ $\quad$ 0 $\quad$ $f\downarrow$ $\quad$ 0 $\quad$ $f\uparrow$ ← + + + + + + + + + + + + + $(-1/2)$---0 ---------- (1)+ + + + + + + + + + + + + + → f' $\cap$ $\quad$ 0 $\quad$ $\cup$ ← --0--- $(1/4)$+ + + + + + + + + + + + + + + + → f''



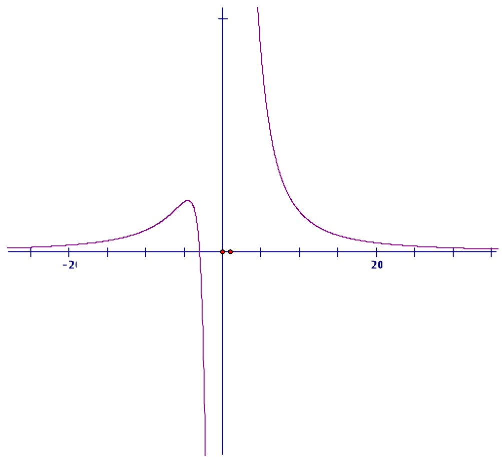
Sketch the graph of $f(x) = \frac{x}{1+x^2}$

$f'(x) = \frac{1-x^2}{(1+x^2)^2}$ $\qquad$ $f''(x) = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(1+x^2)^3}$
← --------------------------------------- 0 + + + + + + + + + + + + + + → f $f\downarrow$ $\quad$ 0 $\quad$ $f\uparrow$ $\quad$ $f\uparrow$ $\quad$ 0 $\quad$ $f\downarrow$ ← --------------- (-1)+ + + + + + + + 0 + + + + + + (1)-------------- → f' $\cap$ $\quad$ 0 $\quad$ $\cup$ $\quad$ 0 $\quad$ $\cap$ $\quad$ 0 $\quad$ $\cup$ ← --------- $(-\sqrt{3})$+ + + + + + + + + 0 -------------- $(\sqrt{3})$+ + + ++ → f''



Sketch the graph of $f(x) = 400\dfrac{x+2}{x^3}$ **.**

$f'(x) = -800\dfrac{(x+3)}{x^4}$ $\qquad$ $f''(x) = 2400\dfrac{(x+4)}{x^5}$
$f > 0$ $\quad$ $f > 0$ $\quad$ $f > 0$ $\quad$ 0 $\quad$ $f < 0$ $\quad$ ND $\quad$ $f > 0$ $\leftarrow$ + + + + + + + + + + + + (-4) + + + + + + (-3) + + + + (-2)--------- 0 + + + + + $\rightarrow$ f $f\uparrow$ $\quad$ $f\uparrow$ $\quad$ 0 $\quad$ $f\downarrow$ $\quad$ ND $\quad$ $f\downarrow$ $\leftarrow$ + + + + + + + + + + + + (-4) + + + + + + (-3)---------- (-2)---------0------------ $\rightarrow$ f' $\cup$ $\quad$ $\cup$ $\quad$ 0 $\quad$ $\cap$ $\quad$ $\cap$ $\quad$ $\cap$ $\quad$ ND $\quad$ $\cup$ $\leftarrow$ + + + + + + + + + + + + (-4)-------------- (-3)---------- (-2)------- 0 + + + + + $\rightarrow$ f''



Antiderivatives

Most mathematical operations have **anti** operations. In algebra, for example, the anti operation of multiplication is division, that of addition is subtraction, and that of raising to a power is taking a root. More commonly, these anti operations in algebra are called inverse operations. As you can see, each anti operation undoes the original operation. In the Calculus, the anti operation of differentiation (finding the derivative) is called **antidifferentiation** or **integration**. The result of an application of antidifferentiation is called an **antiderivative**. For example, since $\left(x^2\right)' = 2x$, we say that x^2 is an antiderivative of $2x$ and since $(\sin x)' = \cos x$, we say that $\sin x$ is an antiderivative of $\cos x$. Please note that antiderivatives are not unique. For example, we also have $\left(\sin x + \sqrt{2}\right)' = \cos x$. So, $\sin x + \sqrt{2}$ is also an antiderivative of $\cos x$.

Definition of an Antiderivative

Let f be continuous on an interval I.
F is an <u>antiderivative</u> of f on I $\overset{def}{\equiv}$ if F is continuous on I and $F' = f$ on the interior of I.

By a previous corollary to the Mean Value Theorem (MVT), if $F' = G'$ on an interval I, then F and G must differ by some constant on I. That is, $G = F + C$ on I, for some constant C. Therefore, if F is an antiderivative of f on I, the most general antiderivative of f on I is given by $F + C$ on I, for some constant C. Since D_x denotes differentiation with respect to x, we will at times suggestively use A_x to denote antidifferentiation with respect to x. Thus, $A_x(f(x)) = F(x) + C$, on I. Even though the notation $A_x(f(x))$ is most reasonable, this is not the notation that is universally used to denote the general antiderivative. This latter notation is reserved for $\int f(x)dx$ or $\int f$, and is due to Leibniz.

Definition of the General Antiderivative or Indefinite Integral

Let f be continuous on an interval I.
We denote by $\int f(x)dx$ the <u>indefinite integral</u> (general antiderivative) of f on I. Then, if C is an arbitrary constant, called the <u>constant of integration</u>,
$\int f(x)dx = F(x) + C \overset{def}{\equiv}$ if $F' = f$ on I.

See Figure 7 for a graphic description of the relationship between integration and differentiation.

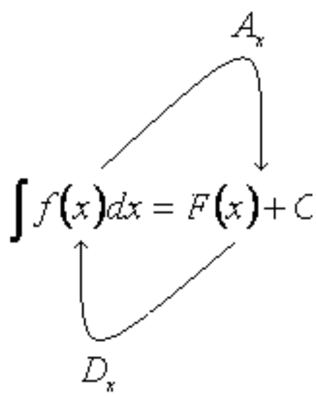


Figure 7

Stated in another way,

$$A_x(f(x)) = F(x) + C \Leftrightarrow D_x(F(x) + C) = f(x)$$

Basic Table of Indefinite Integrals

1. $\int 0\, dx = C$
2. $\int x^r\, dx = \dfrac{x^{r+1}}{r+1} + C,\ -1 \neq r \in \mathbb{R}$
3. $\int \sin x\, dx = -\cos x + C$
4. $\int \cos x\, dx = \sin x + C$
5. $\int \sec^2 x\, dx = \tan x + C$
6. $\int \csc^2 x\, dx = -\cot x + C$
7. $\int \sec x \tan x\, dx = \sec x + C$
8. $\int \csc x \cot x\, dx = -\csc x + C$
9. $\int \dfrac{dx}{x} = \ln|x| + C,\ x \neq 0$
10. $\int e^x\, dx = e^x + C$
11. $D_x \int f(x) dx = f(x)$
12. $\int D_x(F(x)) dx = F(x) + C$
13. $\int f(g(x)) g'(x) dx = (F \circ g)(x) + C$, where $F'(x) = f(x)$

    Proof: $\int f(g(x)) g'(x) dx = \int F'(g(x)) g'(x) dx = \int (F \circ g)'(x) dx = (F \circ g)(x) + C$. $\square$

Linearity of the Indefinite Integral

Let $\alpha, \beta \in \mathbb{R}$. Then

$$\left.\begin{aligned} &1.\ \int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx \\ &2.\ \int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx \\ &3.\ \int \alpha \cdot f(x)\, dx = \alpha \cdot \int f(x)\, dx \end{aligned}\right\} \Leftrightarrow \int [\alpha \cdot f(x) + \beta \cdot g(x)]\, dx = \alpha \cdot \int f(x)\, dx + \beta \cdot \int g(x)\, dx$$

Definition of the Differential

Let $y = f(x)$ on I.
The differential $dy = f'(x)dx$ on $I \overset{def}{\equiv}$ if $\dfrac{dy}{dx} = f(x)$ on I.

Let $F'(x) = f(x)$. Using the Differential, we may write

$$\left.\begin{array}{l} u = g(x) \\ du = g'(x)dx \end{array}\right\} \Rightarrow \int f(g(x))g'(x)dx = \int f(u)du\Big|_{u=g(x)} = F(u) + C\big|_{u=g(x)} = (F \circ g)(x) + C$$

Example:

$$\left.\begin{array}{l} u = (x^3 + 1) \\ du = 3x^2 dx \\ \dfrac{du}{3} = x^2 dx \end{array}\right\} \Rightarrow \int x^2 \cos(x^3 + 1)dx = \frac{1}{3}\int \cos u \, du\Big|_{u=x^3+1} = \frac{1}{3}\sin u + C\big|_{u=x^3+1} = \frac{1}{3}\sin(x^3 + 1) + C$$

Differential Equations

Def: A differential equation is an equation involving the independent variable *x*, the dependent variable *y* and the latter's derivatives expressed as $F\left(x, y, y', y'', \cdots y^{(n)}\right) = 0$. Our goal is to solve such an equation; that is, to find a function $y = f(x)$ such that it and its various derivatives satisfy this equation. We will study a type of differential equation called a first order separable differential equation. This is the type in which only the variables x, y and y' appear and in which the expressions involving the variable *x* can be separated to one side of the equal sign and the expressions involving the variable *y* can be separated to other side of the equal sign. This separation is accomplished most easily by using differentials.

Examples:

(a) Find the *xy*-equation of the curve through $(1,2)$ whose slope at any point is the ratio of its first coordinate to its second coordinate.

Solve: $\frac{dy}{dx} = \frac{x}{y}$

$\frac{dy}{dx} = \frac{x}{y} \underset{\text{separate variables}}{\Rightarrow} ydy = xdx \Rightarrow \int ydy = \int xdx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$. What is *C*? This latter equation implies, using our initial condition, that

$\frac{4}{2} = \frac{1}{2} + C \Rightarrow C = \frac{3}{2}$. Thus, $\frac{y^2}{2} = \frac{x^2}{2} + \frac{3}{2} \Rightarrow y^2 = x^2 + 3$.

Answer: $y = \sqrt{x^2 + 3}$.

(b) Find $y(x)$ such that $\frac{dy}{dx} = \frac{x^2 + x}{y^2}$ and $y(-1) = 3$.

Solve: $\frac{dy}{dx} = \frac{x^2 + x}{y^2} \underset{\text{separate variables}}{\Rightarrow} y^2 dy = \left(x^2 + x\right)dx \Rightarrow \int y^2 dy = \int \left(x^2 + x\right)dx \Rightarrow \frac{y^3}{3} = \frac{x^3}{3} + \frac{x^2}{2} + C$. What is *C*?

Then, as before, $9 = \frac{-1}{3} + \frac{1}{2} + C \Rightarrow C = \frac{53}{6}$.

Thus, $\frac{y^3}{3} = \frac{x^3}{3} + \frac{x^2}{2} + \frac{53}{6} \Rightarrow 2y^3 = 2x^3 + 3x^2 + 53$.

Answer: $y = \sqrt[3]{\frac{2x^3 + 3x^2 + 53}{2}}$.

(c) Find the equation of motion of an object moving vertically near the surface of Planet X with initial velocity v_0 and a height y_0 from the surface. Planet X has an acceleration due to gravity of a units of distance per second per second.

Solve: $\frac{dv}{dt} = a$

$\frac{dv}{dt} = a \underset{\text{separate variables}}{\Rightarrow} dv = adt \Rightarrow \int dv = \int adt \Rightarrow v = at + C_1$. What is C_1? Then, as before,

$v_0 = v(0) = 0 + C_1 \Rightarrow C_1 = v_0$. Thus, so far, $v = at + v_0$. Again,

$\frac{dy}{dt} = v = at + v_0 \underset{\text{separate variables}}{\Rightarrow} dy = (at + v_0)dt \Rightarrow \int dy = \int (at + v_0)dt \Rightarrow y = a\frac{t^2}{2} + v_0 t + C_2$. What is C_2?

Then, as before, $y_0 = y(0) = 0 + 0 + C_2 \Rightarrow C_2 = y_0$.

Answer: $y = a\frac{t^2}{2} + v_0 t + y_0$.

If the object is dropped, then $v_0 = 0$ and the solution becomes $y = a\frac{t^2}{2} + y_0$.

If the object is on the surface of Planet X, then $y_0 = 0$ and the solution becomes $y = a\frac{t^2}{2} + v_0 t$

If Planet X is the Earth, then $a = -32\,\text{ft} / \text{second}^2$.

Remark: To every differentiation rule there corresponds a differential rule. For example, to the product rule there corresponds the differential rule $d(u \cdot v) = u \cdot dv + v \cdot du$.